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Strong Non-Contextual Holism in Quantum Macroscopic States

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To the memory of Rolando Chuaqui (1935-1994)

Abstract

Contextuality is often associated with nonseparability and holism in quantum mechanics. Here we show that some N-particle quantum systems have a set of non-contextual observables that are holistic, such that the system is deterministic, whereas all its parts are random. The total correlation is not sufficient to determine the probability distribution, presenting a need for extra measurements. We propose a formal definition of holism not based on contextuality.

Keywords: contextuality, holism, quantum mechanics

Introduction

Bounded contextuality is perhaps the most important defining characteristic of quantum mechanics [5, 30]. Indeed most of the puzzling aspects of quantum mechanics are related to its contextual character. This is the case for the double slit experiment [6], and for the entanglement of two or more states, as the Einstein-Podolsky-Rosen (EPR) paradox [19, 21] or the Greenberger-Horne-Zeilinger (GHZ) theorem [25, 7]. We call it bounded contextuality because some of those systems exhibit less contextuality than general non-signaling systems [34], as they satisfy what is called the Tsirelson bound [13]. Another case is that of state independent contextuality, as the famous Kochen-Specker theorem [27].

Contextuality is indicated by the lack of a joint probability distribution for a system of observables (see [4, 3, 16, 15] and references therein). Intuitively, we think that observables are contextual when they depend on the context, i.e. on how they are measured and with which other variables they are measure together. Formally, we can express this in terms of random variables. Let $E =$

$\{X_1, \ldots, X_N\}$ be a set of possible observable quantities (we will think of such quantities as quantum observables later on). We assume, for simplicity, that the observables X_i are yes-no questions. Let a context C_i be an experimental condition where some of those quantities are observed simultaneously (but not necessarily all of them). For example, C_1 can be the context where X_1 and X_2 are observed simultaneously, C_2 the context where X_2 and X_3 are observed, and so on.

Let us model the outcomes of E with random variables. Let $\{\mathbf{X}_1, \ldots, \mathbf{X}_N\}$ be a set of N ± 1-valued random variables defined on a probability space $(\Omega, \mathcal{F}, p)$. It is not always possible to model the experimental outcomes of E with this set of random variables for two possible reasons. First, the expectations of X_i change with context, which happens when its measurement outcomes can be thought of as being directly affected by the experimental condition. When this happens, we say the system exhibits *explicit contextuality*, as it is unequivocal in this case that the outcomes of experiments change with context. A more subtle example is when the expectations themselves do not change from context to context, but are inconsistent between different contexts. An example is the well-known Kochen-Specker set of observables, but Suppes-Zanotti [40] provided a simpler example, directly related to Specker's parable of the seer [39]. Let $\{\mathbf{X}, \mathbf{Y}, \mathbf{Z}\}$ be a set of ± 1-valued random variables with zero expectations and perfect anti-correlations $E(\mathbf{XY}) = E(\mathbf{XZ}) = E(\mathbf{YZ}) = -1$. Though the expectations of $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ are the same thought the different contexts, by construction, it is not possible for them to have the same values in different contexts. To see this, imagine $\mathbf{X} = 1$ in the context of $(\mathbf{X}, \mathbf{Y})$. From the first correlation, this implies that $\mathbf{Y} = -1$, which from the third correlation yields $\mathbf{Z} = 1$, which contradicts with the third correlation, which requires the product $\mathbf{XZ}$ to be -1. This logical contradiction arising from the expectations does not allow for a joint probability distribution [1]. Systems like Suppes-Zanotti are said to exhibit *hidden contextuality*[1].

In addition to contextuality, EPR and GHZ show a striking characteristic of quantum mechanics: non-locality, or the context dependency of systems situated far apart from each other. In quantum mechanics, systems that interacted with each other in the past may become entangled, and, even if they are separated by a great distance later on, their properties can be correlated in a way that would evade any attempt to give a classical explanation [8]. This non-local contextuality has as a consequence the nonexistence of a joint probability distribution, and hence of a local hidden-variable theory, that explains the outcome of the experiments [40]. More recently, Mermin [29] showed that

[1]Dzhafarov, Kujala, and collaborators reserve the word contextuality only for what we call here hidden contextuality, and they refer to systems exhibiting explicit contextuality as inconsistently connected [17].

if we allow states with a large number N of particles to be superposed in a way similar to the superposition of particles in the GHZ theorem, then quantum mechanics deviates exponentially with N from the classical case (i.e., one that could be understood by a local hidden-variable).

Kochen-Specker [27], on the other hand, does not require separability. It comes not from observational properties of a state, but from the algebra of observables themselves. Because quantum logic is not Boolean, but forms an orthomodular lattice, Kochen and Specker showed that for a set of observables, all commuting for each experimental context, its is impossible to consistently assign truth values to outcomes of all measurements, if we assume that those truth values are not context-dependent.

The non-local contextuality, i.e. nonseparability, of variables that can account for all the experimental outcomes suggests that quantum mechanics has some holistic characteristic. Holism is the idea that the whole cannot be considered as the sum of its individual parts. The fact that systems far apart are contextual and nonseparable has led some authors to suggest that quantum mechanics has in its core a holistic characteristic [23, 35]. Nonseparability, in the sense used in EPR or GHZ, means that a local hidden-variable theory that predicts the outcome of the experiments is impossible. Of course, nonseparability implies holism, but that the converse is not true is what we show in this article.

To do this, we will first show that a GHZ N-particle quantum mechanical system behaves in a deterministic way, when considered as a whole, but that every proper subsystem behaves in a completely random way. This is done by first showing that any subsystem has maximal entropy, whereas the whole system has entropy zero. Then, we analyze, from a probabilistic point of view, the N-particle GHZ example. We show that quantum mechanics is more restrictive on the subsystems than pure probability considerations, even though, for the particular observables in question, a joint probability distribution exists. Then, we propose a definition of holism that is distinct from the concept of separability (i.e., it is not contextual), and discuss this definition by means of simple examples. Our definition of holism is satisfied by the GHZ quantum mechanical system presented earlier.

1 Quantum Mechanical Holism

Before we analyze the more complicated N-particle case, let us start with an example of a three-particle quantum-entanglement know as the GHZ state [25]. The GHZ state is described by the vector

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left(|+++\rangle - |---\rangle \right), \tag{1}$$

where $|+++\rangle \equiv |+\rangle_{1z} \otimes |+\rangle_{2z} \otimes |+\rangle_{3z}$ is the state where all three particles are prepared as an eigenstate of value $+1$ of the z-spin operator (for simplicity, we are using a system of units where $\hbar = 1$; details about spin operator algebra can be found in standard quantum mechanics textbooks, as for example Cohen-Tanoudji et al. [14] or Sakurai [37]).

From the algebra of spin operators, we have

$$\hat{\sigma}_{zi}|+\rangle_{zi} = |+\rangle_{zi}, \quad \hat{\sigma}_{zi}|-\rangle_{zi} = -|-\rangle_{zi}, \tag{2}$$

$$\hat{\sigma}_{xi}|+\rangle_{zi} = |-\rangle_{zi}, \quad \hat{\sigma}_{xi}|-\rangle_{zi} = |+\rangle_{zi}, \tag{3}$$

$$\hat{\sigma}_{yi}|+\rangle_{zi} = -i|-\rangle_{zi}, \quad \hat{\sigma}_{yi}|-\rangle_{zi} = i|+\rangle_{zi}, \tag{4}$$

where $\hat{\sigma}_{wj}$, $w = x, y, z$ and $j = 1, 2, 3$, represents the w-direction spin observable for particle j.

The state described by (1) has many important properties, stated in the following proposition.

Proposition 1.1 *The state vector (1) is an eigenvector of* $\hat{\sigma}_{x1} \otimes \hat{\sigma}_{y2} \otimes \hat{\sigma}_{y3}$, $\hat{\sigma}_{y1} \otimes \hat{\sigma}_{x2} \otimes \hat{\sigma}_{y3}$, $\hat{\sigma}_{y1} \otimes \hat{\sigma}_{y2} \otimes \hat{\sigma}_{x3}$, *and* $\hat{\sigma}_{x1} \otimes \hat{\sigma}_{x2} \otimes \hat{\sigma}_{x3}$, *and and its eigenvalues are, respectively,* 1, 1, 1, *and* -1.

Proof. We start with $\hat{\sigma}_{x1} \otimes \hat{\sigma}_{y2} \otimes \hat{\sigma}_{y3}$. In order to compute the $\hat{\sigma}_{x1} \otimes \hat{\sigma}_{y2} \otimes \hat{\sigma}_{y3}|\psi\rangle$ we first compute $\hat{\sigma}_{x1} \otimes \hat{\sigma}_{y2} \otimes \hat{\sigma}_{y3}|+++\rangle$ and $\hat{\sigma}_{x1} \otimes \hat{\sigma}_{y2} \otimes \hat{\sigma}_{y3}|---\rangle$ using (3) and (4).

$$\begin{aligned}
\hat{\sigma}_{x1} \otimes \hat{\sigma}_{y2} \otimes \hat{\sigma}_{y3}|+++\rangle &= \hat{\sigma}_{x1}|+\rangle_{z1} \otimes \hat{\sigma}_{y2}|+\rangle_{z2} \otimes \hat{\sigma}_{y3}|+\rangle_{z3} \\
&= |-\rangle_{z1} \otimes (-i)\,|-\rangle_{z2} \otimes (-i)\,|-\rangle_{z3} \\
&= -|-\rangle_{z1} \otimes |-\rangle_{z2} \otimes |-\rangle_{z3} \\
&= -|---\rangle,
\end{aligned}$$

$$\begin{aligned}
\hat{\sigma}_{x1} \otimes \hat{\sigma}_{y2} \otimes \hat{\sigma}_{y3}|---\rangle &= \hat{\sigma}_{x1}|-\rangle_{z1} \otimes \hat{\sigma}_{y2}|-\rangle_{z2} \otimes \hat{\sigma}_{y3}|-\rangle_{z3} \\
&= -|+\rangle_{z1} \otimes i|+\rangle_{z2} \otimes i|+\rangle_{z3} \\
&= |+\rangle_{z1} \otimes |+\rangle_{z2} \otimes |+\rangle_{z3} \\
&= |+++\rangle.
\end{aligned}$$

Therefore

$$\begin{aligned}
\hat{\sigma}_{x1} \otimes \hat{\sigma}_{y2} \otimes \hat{\sigma}_{y3}|\psi\rangle &= \frac{1}{\sqrt{2}}\left(\hat{\sigma}_{x1} \otimes \hat{\sigma}_{y2} \otimes \hat{\sigma}_{y3}|+++\rangle - \hat{\sigma}_{x1} \otimes \hat{\sigma}_{y2} \otimes \hat{\sigma}_{y3}|---\rangle\right) \\
&= \frac{1}{\sqrt{2}}\left(-|---\rangle + |+++\rangle\right) \\
&= |\psi\rangle.
\end{aligned}$$

as we wished to prove. The proofs for $\hat{\sigma}_{y1} \otimes \hat{\sigma}_{x2} \otimes \hat{\sigma}_{y3}$, $\hat{\sigma}_{y1} \otimes \hat{\sigma}_{y2} \otimes \hat{\sigma}_{x3}$, and $\hat{\sigma}_{x1} \otimes \hat{\sigma}_{x2} \otimes \hat{\sigma}_{x3}$ are similar and will be omitted. ■

We are now in a position to state GHZ's main result.

Proposition 1.2 (GHZ) *Let $|\psi\rangle$ be the quantum mechanical state shown in (1) and let $\mathbf{X}_i$ ($\mathbf{Y}_i$) be ± 1 random variables representing the values of the spin in the x-direction (y-direction) for particle i. Then, there exists no joint probability distribution for $\mathbf{X}_i$ and $\mathbf{Y}_i$ that reproduces the quantum mechanical predictions given in Proposition 1.*

Proof. We prove it by showing that the existence of a joint probability distribution leads to a contradiction. From Proposition 1

$$\begin{aligned} 1 &= E(\mathbf{X}_1\mathbf{Y}_2\mathbf{Y}_3) && (5)\\ &= E(\mathbf{Y}_1\mathbf{X}_2\mathbf{Y}_3) && (6)\\ &= E(\mathbf{Y}_1\mathbf{Y}_2\mathbf{X}_3), && (7) \end{aligned}$$

and

$$E(\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3) = -1.$$

If we assume that a joint probability distribution exists, then

$$E((\mathbf{X}_1\mathbf{Y}_2\mathbf{Y}_3)(\mathbf{Y}_1\mathbf{X}_2\mathbf{Y}_3)(\mathbf{Y}_1\mathbf{Y}_2\mathbf{X}_3))$$

exists. From (5)–(7) we have that

$$\begin{aligned} E((\mathbf{X}_1\mathbf{Y}_2\mathbf{Y}_3)(\mathbf{Y}_1\mathbf{X}_2\mathbf{Y}_3)(\mathbf{Y}_1\mathbf{Y}_2\mathbf{X}_3)) &= E((1)(1)(1))\\ &= 1. && (8) \end{aligned}$$

But from the existence of a joint and from the property that a ± 1 random variable square is always equal to 1 we have that

$$\begin{aligned} E((\mathbf{X}_1\mathbf{Y}_2\mathbf{Y}_3)(\mathbf{Y}_1\mathbf{X}_2\mathbf{Y}_3)(\mathbf{Y}_1\mathbf{Y}_2\mathbf{X}_3)) &= E(\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3\mathbf{Y}_1^2\mathbf{Y}_2^2\mathbf{Y}_3^2)\\ &= E(\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3)\\ &= -1, \end{aligned}$$

in clear contradiction to (8), as we wished to prove. ■

The GHZ state has the additional interesting characteristic that combinations of spin measured in different particles behave deterministically. To explain what we mean, let $\mathbf{X}_1$ be a ± 1-valued random variable representing a measurement of spin in the x-direction for particle 1, and let $\mathbf{Y}_2$ and $\mathbf{Y}_3$

be ± 1-valued random variables representing measurements of spin in the y-direction for particle 2 and 3, respectively. If we compute the expectations of these random variables we obtain at once

$$\begin{aligned} E(\mathbf{X}_1) &= \langle\psi|\hat{\sigma}_{x1}|\psi\rangle \\ &= 0, \end{aligned}$$

$$E(\mathbf{Y}_2) = 0,$$

and

$$E(\mathbf{Y}_3) = 0.$$

It is also easy to show that the expectations $E(\mathbf{X}_1\mathbf{Y}_2) = E(\mathbf{X}_1\mathbf{Y}_3) = E(\mathbf{Y}_2\mathbf{Y}_3)$ are zero. However, the product of all three variables, $\mathbf{X}_1\mathbf{Y}_2\mathbf{Y}_3$ is always 1, a deterministic result. We thus conclude that for the three-particle GHZ state, we can define random variables such that their product behaves deterministically, whereas if we remove only one of them, the product of the remaining variables behave randomly.

It is interesting to note that, even though the results above are derived from quantum mechanical considerations, they could be explained classically, i.e., in terms of local hidden variables. The reason is that for the set of observable expectations given are not contextual, in the sense that there is a joint probability distribution. As we know, the existence of a joint probability distribution is a necessary and sufficient condition for the existence of a hidden variable that factors out the observed correlations [40, 21]. Within the GHZ setup, to obtain contextuality one needs to include further experimental conditions (contexts) that yield contradictions.

The consideration above motivates us to extend the three-particle GHZ state to N particles. Let $|\psi\rangle$ be the entangled GHZ-like N-particle state given by

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left[\prod_{k=1}^{N}|+\rangle_k + \prod_{k=1}^{N}|-\rangle_k\right], \tag{9}$$

where $\widehat{\sigma}_{iz}|+\rangle_i = |+\rangle_i$, $\widehat{\sigma}_{iz}|-\rangle_i = -|-\rangle_i$, with $\widehat{\sigma}_{iz}$ being the spin operator in the z direction acting on the i-th particle, $i = 1, \ldots, N$. We start with the following Proposition.

Proposition 1.3 *Given the ket*

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|++\cdots+\rangle + |--\cdots-\rangle), \tag{10}$$

and the operator $\hat{\Sigma} = \widehat{\sigma}_{1x} \otimes \widehat{\sigma}_{2x} \otimes \cdots \otimes \widehat{\sigma}_{Nx}$*, where* $\hat{\sigma}_{ix}$ $(i = 1 \ldots N)$ *is the spin operator corresponding to the observable for spin of the i-th particle in the x direction, then* $|\psi\rangle$ *is an eigenstate of* $\hat{\Sigma}$ *with eigenvalue* 1.

Proof. First, we recall that $\hat{\sigma}_{ix}|+\rangle_i = |-\rangle_i$ and $\hat{\sigma}_{ix}|-\rangle_i = |+\rangle_i$. We can write $\hat{\Sigma}$ in a compact way as

$$\hat{\Sigma} = \prod_{k=1}^{N} \hat{\sigma}_{k,x}. \tag{11}$$

Therefore

$$\begin{aligned}
\hat{\Sigma}|\psi\rangle &= \frac{1}{\sqrt{2}} \prod_{k=1}^{N} \hat{\sigma}_{k,x} \left(\prod_{k=1}^{N} |+\rangle_k + \prod_{k=1}^{N} |-\rangle_k \right) \\
&= \frac{1}{\sqrt{2}} \prod_{k=1}^{N-1} \hat{\sigma}_{k,x} \left(\prod_{k=1}^{N-1} |+\rangle_k \otimes |-\rangle_N + \prod_{k=1}^{N-1} |-\rangle_k \otimes |+\rangle_N \right) \\
&= \frac{1}{\sqrt{2}} \prod_{k=1}^{N-n} \hat{\sigma}_{k,x} \left(\prod_{k=1}^{N-n} |+\rangle_k \otimes \prod_{l=N-n+1}^{N} |-\rangle_l + \prod_{k=1}^{N-n} |-\rangle_k \otimes \prod_{l=N-n+1}^{N} |+\rangle_l \right) \\
&= \frac{1}{\sqrt{2}} \left(\prod_{l=1}^{N} |-\rangle_l + \prod_{l=1}^{N} |+\rangle_l \right) \\
&= |\psi\rangle,
\end{aligned}$$

as we wished to prove. ■

In other words, the observable $\hat{\Sigma}$, made out of the product of all N spin observables, is deterministic, as a measurement of it always results in the value 1. In a similar way, this determinism is also true for the observables

$$\prod_{i} \widehat{\sigma}_{iy} \otimes \prod_{j} \widehat{\sigma}_{jx}, \tag{12}$$

where the index i is any subset with even cardinality of $2^{\{1,2,\ldots,N\}}$, and j is the complement of i.

The state (9) has been the focus of several interesting papers, all of them related to the deterministic aspects of the above observables [25, 29, 10, 28, 9, 7, 41, 31]. However, in this paper we will be interested in observables acting only on a subset of the set of all particles in (9). We start with the following.

Proposition 1.4 *Given the ket*

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|++\cdots+\rangle + |--\cdots-\rangle), \tag{13}$$

and the spin operators $\widehat{\sigma}_{id}$, where $i = 1\ldots N$ and $d = x, y$, then any product of $n < N$ distinct spin operators has expectation zero for this ket. Furthermore, if we include spin operators in the z direction, any product involving an odd number of $\hat{\sigma}_{iz}$'s also has expectation zero.

Proof. Let us call A the set $\{1,2,3,\ldots,N\}$, and let X,Y, Z, and W be disjoint sets such that $X\cup Y\cup Z\cup W = A$. Then, any product of spin operators can be written as

$$\hat{\Sigma}_A = \prod_{k\in X}\hat{\sigma}_{k,x}\otimes\prod_{l\in Y}\hat{\sigma}_{k,y}\otimes\prod_{m\in Z}\hat{\sigma}_{k,z}\otimes\prod_{n\in W}\hat{1}_k.$$

We want to compute $\langle\psi|\hat{\Sigma}_A|\psi\rangle$, the expected value of this operator.

$$\begin{aligned}
\hat{\Sigma}_A|\psi\rangle &= \frac{1}{\sqrt{2}}\prod_{k\in X}\hat{\sigma}_{k,x}\otimes\prod_{l\in Y}\hat{\sigma}_{k,y}\otimes\prod_{m\in Z}\hat{\sigma}_{k,z}\otimes\prod_{n\in W}\hat{1}_k\left(\prod_{k=1}^{N}|+\rangle_k+\prod_{k=1}^{N}|-\rangle_k\right)\\
&= \frac{1}{\sqrt{2}}\prod_{k\in X}\hat{\sigma}_{k,x}\otimes\prod_{l\in Y}\hat{\sigma}_{k,y}\otimes\prod_{m\in Z}\hat{\sigma}_{k,z}\\
&\quad\otimes\prod_{n\in W}\hat{1}_k\left(\prod_{k\in X,Y,Z,W}|+\rangle_k+\prod_{k\in X,Y,Z,W}|-\rangle_k\right)\\
&= \Bigg[(-i)^{C(Y)}\prod_{k\in X,Y}|-\rangle_k\prod_{l\in Z,W}|+\rangle_l\\
&\quad+(-1)^{C(Z)}\,i^{C(Y)}\prod_{k\in X,Y}|+\rangle_k\prod_{l\in Z,W}|-\rangle_l\Bigg],
\end{aligned}$$

where we used $\hat{\sigma}_{ky}|+\rangle_k = -i|-\rangle_k$ and $\hat{\sigma}_{ik}|-\rangle_k = i|+\rangle_k$, and $C(Y)$ ($C(Z)$) is the number of elements of Y (Z). We then have

$$\begin{aligned}
\langle\psi|\hat{\Sigma}_A|\psi\rangle &= \frac{1}{2}\left[\prod_{k=1}^{N}\langle+|_k+\prod_{k=1}^{N}\langle-|_k\right]\times\\
&\quad\Bigg[(-i)^{C(Y)}\prod_{k\in X,Y}|-\rangle_k\prod_{l\in Z,W}|+\rangle_l\\
&\quad+(-1)^{C(Z)}\,i^{C(Y)}\prod_{k\in X,Y}|+\rangle_k\prod_{l\in Z,W}|-\rangle_l\Bigg].
\end{aligned}\tag{14}$$

From the equation above, it is immediate that the inner product is zero if $1<C(Y)+C(X)<N$. Also, it is clear that $\langle\psi|\hat{\Sigma}_A|\psi\rangle$ can be different from zero only if $C(Z)\neq 0$ is even *and* $C(X)=C(Y)=0$, which concludes the proof. ■

Proposition 3 and 4 shows that the correlations for the N-particle system are quite strange. We have a set of N particles that has always the same observable associated to its totality, but when we look at any of its parts, then the parts are completely uncorrelated (say, for the x-direction spins). In this system the presence of a nonzero correlation appears only when we look at the system as a whole, and not at its parts. In the next section we will analyze in details the properties of the probability distribution associated to, say, the operator $\hat{\Sigma}$.

2 Probabilistic Properties

It is interesting to note the consequences of the previous result. Say we are measuring the spin in the x direction for $n < N$ particles. In this case all the particles are independent, and also behave in a completely random way, as the probability of measuring 1 is the same as the probability of measuring -1. However, if we measure the spin of *all* N particles, the whole system is deterministic in a sense that will be made clear later. First, let us start with the following Proposition.

Proposition 2.1 *Let*

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|++\cdots+\rangle + |--\cdots-\rangle), \tag{15}$$

$\hat{\Sigma} = \prod_{k=1}^{N} \hat{\sigma}_{k,x}$, *and to each particle* i, $1 \leq i \leq N$, *we associate the random variable* $\mathbf{S}_i$, *representing the value of its spin measurements, taking values* ± 1. *If* $t = n\Delta t$, $n = 0, 1, 2, \ldots$ *and we measure* $|\psi\rangle$ *using* $\hat{\Sigma}$ *at each* t. *We define the random variables* $\mathbf{X}_t^{\{k\}} = \prod_{\{k\}} \mathbf{S}_k$, *where* $\{k\}$ *is any proper subset of* $\{1, \ldots, N\}$ *and* $\mathbf{X}_t = \prod_{k=1}^{N} \mathbf{S}_k$. *Then each* $\mathbf{X}_t^{\{k\}}$, *and* $\mathbf{X}_t$ *define Bernoulli processes.*

Proof. First we should note that $|\psi\rangle$ is an eigenstate of $\hat{\Sigma}$, such that we can measure $\hat{\Sigma}$ as many times as we want without affecting $|\psi\rangle$. Second, we should note that none of the individual particles i are eigenstates of the spin operator $\hat{\sigma}_{k,x}$, and a measurement of a single particle behaves in a way isomorphic to the tossing of a coin. If we keep measuring spin in the x direction for all particles in equal intervals of time Δt, we can make a data table for the experimental result that would look like Table 1, where we associate to each of the spin measurements for particle i the random variable $\mathbf{S}_i$ taking values ± 1.

Each column of this table would be completely uncorrelated to the any other column or combinations of columns with less than N columns involved. Similar independence and randomness hold for any row of length at most $N-1$, i.e., at least one entry is deleted. However, if we multiply $\mathbf{S}_1$, $\mathbf{S}_2$,$\cdots$, $\mathbf{S}_N$, we

	$\mathbf{S}_1$	$\mathbf{S}_2$	$\cdots$	$\mathbf{S}_N$	$\prod_{i=1}^N \mathbf{S}_i$
0	1	−1	$\cdots$	1	1
Δt	1	−1	$\cdots$	−1	1
$2\Delta t$	−1	1	$\cdots$	1	1
$3\Delta t$	−1	−1	$\cdots$	−1	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 1: Possible set of experimental data results for the random variables $\mathbf{S}_1$, $\mathbf{S}_2,\cdots$, $\mathbf{S}_N$, and $\prod_{i=1}^N \mathbf{S}_i$.

always obtain the same value $\prod_{i=1}^N \mathbf{S}_i = 1$. Furthermore, since the wave function $|\psi\rangle$ is unchanged, the equal probabilities of obtaining a 1 or -1 for each of the columns or shortened rows are also unchanged. As a consequence, the temporal sequence of product random variables $\mathbf{X}_t^{\{k\}} = \prod_{\{k\}} \mathbf{S}_k$, where $\{k\}$ is any proper subset of $\{1,\ldots,N\}$, form a Bernoulli process, i.e. at each time t the random variables $\mathbf{X}_t^{\{k\}}$ are independently and identically distributed, as we wanted to show. It is straightforward to extend the same argument to $\mathbf{X}_t$. ■

We are now in a position to make explicit the statement that the system as a whole is deterministic and its subsystems are random.

Proposition 2.2 *The random variables $\mathbf{X}_t^{\{k\}} = \prod_{\{k\}} \mathbf{S}_k$, where $\{k\}$ is any proper subset of $\{1,\ldots,N\}$, defined in a way similar to Proposition 5, have maximal entropy for such process, whereas the random variable $\mathbf{X}_t = \prod_{k=1}^N \mathbf{S}_k$ has zero entropy.*

Proof. Since both $\mathbf{X}_t^{\{k\}}$ and $\mathbf{X}_t$ define a Bernoulli process, their entropy is $H = -\sum p_i \log p_i$, where p_i is the probability of each possible outcome, in this case ± 1. $\mathbf{X}_t = \prod_{i=1}^N \mathbf{S}_i$, representing the system as a whole, has entropy zero, since for all t $P(\mathbf{X}_t = 1) = 1$ and $P(\mathbf{X}_t = -1) = 0$. Yet, any proper subset $\{k\}$ of $\{1,\ldots,N\}$ will define a random variable $\mathbf{X}_t^{\{k\}} = \prod_{\{k\}} \mathbf{S}_k$ whose entropy is maximal for such a process, as $P(\mathbf{X}^{\{k\}} = 1) = 1/2$ and $P(\mathbf{X}^{\{k\}} = -1) = 1/2$, i.e. the entropy $H = -\sum p_i \log p_i = 1$, where log is to base 2, as we wanted to prove. ■

The results just obtained show that the system in question is strongly holistic, in the sense that a measurement of $\hat{\Sigma}$ containing *all* particles in the system yields a deterministic result, whereas any spin measurement made on a subsystem has a perfectly random outcome. However, since we can measure all the N spin values simultaneously, we can also write a data table for the experimental outcomes, and a joint probability distribution exists. In this sense, the system is holistic but is non-contextual, as we can factor the joint probability distribution.

Even though a joint probability distribution exists, we stress that such a strange distribution, where only when we consider all particles is the system deterministic, is rarely if ever found in any classical empirical domain. In fact, quantum mechanics provides, as far as we know, the *only* example in nature of a case where we have perfect correlation for a triple and zero correlation for pairs (or, for the N-particle case, perfect correlation for the N-th moment and zero for any N'-th moment, where $N' < N$). This is the case if we take a three-particle GHZ system, as it yields $\mathbf{X}_i$ ± 1 random variables, with $E(\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3) = 1$, $E(\mathbf{X}_i) = 0$, $i = 1, \ldots, 3$. It is also interesting to stress that, in the three-particle GHZ case, the pair correlations are zero as a consequence of the triple correlation and the individual expectations. This can be verified by direct computation. Say we have $E(\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3) = 1$. Then, all terms with 0 or 2 negative components sum to 1, i.e.,

$$x_1x_2x_3 + \bar{x}_1\bar{x}_2x_3 + \bar{x}_1x_2\bar{x}_3 + x_1\bar{x}_2\bar{x}_3 = 1, \tag{16}$$

where we use the notation x_1 to represent $P(\mathbf{X}_1 = 1)$, $\bar{x}_1$ to represent $P(\mathbf{X}_1 = -1)$, $\bar{x}_1x_2$ to represent $P(\mathbf{X}_1 = -1, \mathbf{X}_2 = 1)$, and so on. We also have that

$$\begin{aligned} x_1x_2 = x_1x_2x_3 = x_1x_3 = x_2x_3 &= a, && (17)\\ \bar{x}_1\bar{x}_2 = \bar{x}_1\bar{x}_2x_3 = \bar{x}_1x_3 = \bar{x}_2x_3 &= b, && (18)\\ \bar{x}_1x_2 = \bar{x}_1x_2\bar{x}_3 = \bar{x}_1\bar{x}_3 = x_2\bar{x}_3 &= c, && (19)\\ x_1\bar{x}_2 = x_1\bar{x}_2\bar{x}_3 = x_1\bar{x}_3 = \bar{x}_2\bar{x}_3 &= d, && (20) \end{aligned}$$

with $a+b+c+d = 1$. Next, from (17)–(20), $x_1 = a+d$, $\bar{x}_1 = b+c$, $x_2 = a+c$, $\bar{x}_2 = b+d$, $x_3 = a+b$, $\bar{x}_3 = c+d$, and from $E(\mathbf{X}_i) = 0$, $x_1 = x_2 = \bar{x}_1 = \bar{x}_2 = \frac{1}{2}$. From (17)–(20) and the following equations, we obtain at once $a = b = c = d$ and

$$E(\mathbf{X}_1\mathbf{X}_2) = E(\mathbf{X}_2\mathbf{X}_3) = E(\mathbf{X}_1\mathbf{X}_3) = 0. \tag{21}$$

However, contrary to the three-particle case, if we increase the number of particles to four, the correlations are not dictated by $E(\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3\mathbf{X}_4) = 1$, $E(\mathbf{X}_i) = 0$, $i = 1, \ldots, 4$ anymore. For the four-particle case, we can compute, in a manner similar to the three-particle one, that $E(\mathbf{X}_i\mathbf{X}_j\mathbf{X}_k) = 0$, $i < j < k$. However, the correlations $E(\mathbf{X}_i\mathbf{X}_j)$ can individually, but not independently, take any value in the closed interval $[-1, 1]$. On the other hand, if all the correlations are zero, then the positive atomic events have a uniform distribution, by an argument similar to the one given above. In fact, we can show the following.

Proposition 2.3 *Given $E(\mathbf{X}_1 \cdots \mathbf{X}_n) = 0$ and the product of any nonempty subset of the random variables $\mathbf{X}_1 \cdots \mathbf{X}_n$ also has expectation zero, including $E(\mathbf{X}_i) = 0$, $1 \leq i \leq n$. Then the 2^n atoms of the probability space supporting $\mathbf{X}_1 \cdots \mathbf{X}_n$ has a uniform probability distribution, i.e., each atom has probability $1/2^n$.*

Proof. We show this by induction. For $n = 1$, we have by hypothesis that $E(\mathbf{X}_i) = 0$, so, as required, $P(\mathbf{X}_i = 1) = x_1 = 1/2$. Next, our inductive hypothesis is that for every subsystem having $m < n$, the 2^m atoms have a uniform distribution, and we need to show this holds for n. Using the induction hypothesis for $n-1$, we have at once the following pair of equations:

$$\begin{aligned} x_1 x_2 \cdots x_{n-1} &= x_1 x_2 \cdots x_{n-1} x_n + x_1 x_2 \cdots x_{n-1} \bar{x}_n = 2^{1-n}, \\ x_1 x_2 \cdots x_{n-2} x_n &= x_1 x_2 \cdots x_{n-1} x_n + x_1 x_2 \cdots \bar{x}_{n-1} x_n = 2^{1-n}. \end{aligned}$$

Subtracting one equation from the other we have

$$x_1 x_2 \cdots \bar{x}_{n-1} x_n = x_1 x_2 \cdots x_{n-1} \bar{x}_n.$$

By similar arguments, we show that all atoms that have exactly one negative value of $\bar{x}_i$ for the n-particle case are equal in probability. Moreover, without any new complication this argument extends to equal probability for any atomic event having exactly k negative values, $2 \leq k \leq n$.

Next, we can easily show that those atoms differing by 2, and therefore by an even number of, negative values have equal probability. We give the argument for $k = 0$ and $k = 2$:

$$\begin{aligned} x_1 x_2 \cdots x_{n-1} &= x_1 x_2 \cdots x_{n-1} x_n + x_1 x_2 \cdots x_{n-1} \bar{x}_n = 2^{1-n}, \\ \bar{x}_1 x_2 \cdots x_{n-2} x_n &= \bar{x}_1 x_2 \cdots x_{n-1} x_n + \bar{x}_1 x_2 \cdots \bar{x}_{n-1} x_n = 2^{1-n}. \end{aligned}$$

Using the previous result and subtracting we get

$$x_1 x_2 \cdots x_{n-1} x_n = \bar{x}_1 x_2 \cdots x_{n-1} \bar{x}_n.$$

Finally, we use the hypothesis that $E(\mathbf{X}_1 \cdots \mathbf{X}_n) = 0$. This zero expectation requires that the sum of all the terms with 0 or an even number of negative values have the same sum as all the terms with an odd number of negative values. This implies at once that all atoms have equal probability, and so each has probability $1/2^n$, proving Proposition 7. ■

We also prove a more restricted result, but a significant one, by purely probabilistic means, i.e., no quantum mechanical concepts or assumptions are needed in the proof.

Proposition 2.4 *Given* $E(\mathbf{X}_1 \ldots \mathbf{X}_N) = \pm 1$ *and* $E(\mathbf{X}_i) = 0$, $i = 1, \ldots, N$, *then any* $(N-1)$*-th moment is zero, e.g.:*
$E(\mathbf{X}_1 \ldots \mathbf{X}_{N-1}) = 0$, $E(\mathbf{X}_1 \ldots \mathbf{X}_{N-2}\mathbf{X}_N) = 0$, *etc.*

Proof. We give the proof for $E(\mathbf{X}_1 \ldots \mathbf{X}_N) = 1$. Then there are 2^N atoms in the probability space. Given the expectation equal to 1, half of the atomic events must have probability 0, namely all those representing negative spin products. Now, we consider all the terms expressing $E(\mathbf{X}_1 \ldots \mathbf{X}_{N-1})$. On the positive side, we have all those with even or zero negative values:

$$x_1 x_2 \cdots x_{N-1} + \bar{x}_1 \bar{x}_2 \cdots x_{N-1} + \cdots + \bar{x}_1 \bar{x}_2 \cdots \bar{x}_{N-1} \tag{22}$$

if $N-1$ is even and as the last term if $N-1$ is odd $x_1 \bar{x}_2 \cdots \bar{x}_{N-1}$. To be extended to atoms, a positive x_N must be added. So, in probability

$$x_1 x_2 \cdots x_{N-1} = x_1 x_2 \cdots x_{N-1} x_N,$$

because, given $E(\mathbf{X}_1 \ldots \mathbf{X}_N) = 1$

$$x_1 x_2 \cdots x_{N-1} \bar{x}_N = 0,$$

and similar for the other terms in (22).

The same thing applies in similar fashion to the negative side, e.g.,

$$\bar{x}_1 x_2 \cdots x_{N-1} = \bar{x}_1 x_2 \cdots x_{N-1} \bar{x}_N,$$

since the atom on the right must have zero or an even number of negative values.

But we observe that, by hypothesis, $E(\mathbf{X}_1 \ldots \mathbf{X}_{N-2}\mathbf{X}_N) = 0$, but the probability x_N is just equal to the sum of the probabilities of the positive terms of $E(\mathbf{X}_1 \ldots \mathbf{X}_{N-2}\mathbf{X}_N)$ and $\bar{x}_N$ is just equal to the sum of the probabilities of the negative terms above. Since, $x_N - \bar{x}_N = 0$, we conclude $E(\mathbf{X}_1 \ldots \mathbf{X}_{N-1}) = 0$. The same argument can be extended to the other $N-1$ combinations of $\mathbf{X}_i$, and this completes the proof. ■

3 Π-Holism

The remarkable property that a quantum system has a perfect correlation for its whole but a totally random behavior for *any* of its part seems to us to represent a holistic characteristic of quantum mechanics. This holism is, however, quite distinct from separability or contextuality. For that reason, we propose the following definition for strict holism.

Definition 3.1 *Let* $(\Omega, \mathcal{F}, p)$ *be a finite probability space and let*

$$\mathbf{F} = \{\mathbf{X}_i, 1 \leq i \leq N\}$$

be a family of ± 1 *random variables defined on* Ω*. Let* Π *be a property defined for finite families of random variables. Then* $\mathbf{F}$ *is strictly* Π*-holistic iff*
(i) $\mathbf{F}$ *has* Π*;*
(ii) *No subfamily of* $\mathbf{F}$ *has* Π*.*
Moreover, if Π *is a numerical property,*
(iii) *No subfamily of* $\mathbf{F}$ *approximates* Π*.*

To understand this definition, let us give some examples from classical mechanics. It is well know in classical gravitation theory that a two-particle system has a well defined solution. However, if we add to this system an extra particle, no closed solutions to this system exist in some cases, and in fact its behavior can be completely random [2]. One may be tempted to think that this chaotic behavior is a holistic property, but according to the definition above, it is not. For instance, let us take the restricted three-body problem analyzed by Alekseev, where two particles with large mass orbit around their Center of Mass (CM), while a third small particle oscillates in a line passing through the CM and perpendicular to the plane of orbit of the two large masses. The whole system behaves randomly, as well at least one subsystem, the one defined by the small particle. Hence, this system is not Π-holistic, if we choose Π to be the property of being random.

As yet another example, let us consider a glass of water. The water is a large system that does not behave like a water molecule, but in a coordinated way dictated by hydrodynamics. Is then this system holistic? If we take, say, half the glass of water, the properties of this half of water are the same as the whole glass, except its mass, hence the system is not Π-holistic for the other macroscopic properties of the water. What about properties like, say, mass? Say we take the full glass and remove only a water molecule from it. The new subsystem approximates the mass of the original one, violating hypothesis **(iii)** from the Definition, and so if we choose Π to be the property mass, the system is not Π-holistic.

Proposition 3.2 *Let* $\mathbf{F} = \{\mathbf{S}_i, i = 1, \ldots, N\}$ *be the set of random variables of all the spin measurements of the state*

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|++\cdots+\rangle + |--\cdots-\rangle),$$

and let $\mathbf{X}_t$ *be the product random variable of Proposition 6, and let* $\mathbf{X}_t^{\{k\}}$ *be the product random variable of any subfamily* $\{k\}$ *as defined earlier. Let the* *entropy be the* Π *property of these product random variables. Then* $\mathbf{F}$ *is* Π-*holistic.*

Proof. Immediate, from Proposition 6, since the entropy of $\mathbf{X_t}$ is 0 and, for any $\{k\}$, the entropy of $\mathbf{X_t}^{\{k\}}$ is 1. ■

To summarize, we found that an N-particle GHZ state has a strong holistic property. However, it may be difficult to detect experimentally a quantum mechanical holistic characteristic with a large number of particles, as decoherence may play an important role, given that the decoherence time decreases rapidly if we increase the number of particles [11, 32, 42]. A promising setup where this holism could be verified for a reasonably large number of particles is the one proposed by Cirac and Zoller [12, 22]. We found that for $N \geq 4$, the measurements of $E(\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3 \cdots \mathbf{X}_N)$ and of $E(\mathbf{X}_i)$ do not fix a probability distribution, and extra measurements are necessary for the pairs, triples, and so on, for the probability distribution to be fixed. We believe that these measurements, which should yield many zero correlations, could be used to put additional constraints on some local-hidden variable models that exploit the detection loophole [18, 33, 38, 24].

We could also remark that this type of holism is not a characteristic only of separable N-particle systems. It is possible have a localized quantum system whose Hilbert space has a large enough dimension to allow for observables that have the same properties as the $\mathbf{X}_1, \ldots, \mathbf{X}_N$ of the N-particle GHZ. Though such systems do not have the same difficulties as multi-particle systems, insofar as decoherence increases with the number of particles, the complexity of the different experimental outcomes may make it hard, as is the case with experimental verifications of the Kochen-Specker theorem [26].

4 Quine's Holism About Language

An important and prominent advocate of holism about language is Quine's view on the dogma of reductionism in his well-known article "Two Dogmas of Empiricism". Here is a succinct statement that states his thesis about the holism of language without, in fact, using the term itself.

> The dogma of reductionism survives in the supposition that each statement, taken in isolation from its fellows, can admit of confirmation or infirmation at all. My countersuggestion, issuing essentially from Carnap's doctrine of the physical world in the *Aufbau*, is that our statements about the external world face the tribunal of sense experience not individually but only as a corporate body. Quine, 1953, p. 41

In a footnote to this passage, Quine refers to similar arguments of Duhem, published much earlier, about physics, rather than language. (This reference to Duhem was added in the revised edition of the paper, published in [36].)

It does not take much reflection to see that the very specific kind of radical or strict holism we are characterizing in this article has a definitely more restricted form than what Quine is proposing about language. For example, one would scarcely say that dropping some single sentence from the corpus of language would radically change the nature of that language. Saying that doesn't argue against the holism of language in the sense Quine is arguing, but, rather, argues against the strict holism defined above. Language is like water, in the sense that removing a single water molecule from a glass of water does not change any of the standard microscopic properties of the water, but it is exactly such a radical change that is characteristic of the quantum holism we have characterized in this article.

Someone might say, well, if we had a very tight formal system of language, a removal of one sentence could upset various closure conditions and inferential relations. But this is not the true nature of language. This is language at its artificial best. Now, we might say the same thing about the quantum-mechanical system we propose. It is capable of explicit physical realization, but it is highly special. Perhaps, therefore, a better way to put the matter is this. We conceive of some rich, artificial language and now remove one sentence. What is the effect of this? This is not something ordinarily studied in the structures of formal languages, because we don't have, for such structures, the kind of computations we've exhibited for entangled particles in quantum mechanics. One possible response, finitistic in character, is that, in any actual use of a formal language, for example, in developing in a semi-formal way, axiomatic set theory, we actually use only a rather restricted finite set of formulas or sentences. We could remove any one sentence and, undoubtedly, find a close enough form to it not to have missed that particular sentence. This is a rather artificial-sounding way of getting around the problem, but, perhaps, enough has been said to make reasonably clear that holism, as Quine thinks about it for language, is a much less well-defined concept, or a much broader concept, if we do not want to quarrel about definition, than the strict holism for quantum mechanics we have considered here.

What we have said about language applies also to the web of beliefs, a closely-related thesis, often advocated by Quine and others. Rather than webs of belief, we would prefer to refer to associative networks, but this is not important. We would accept that there is a kind of weak holism about the web of belief, but it is not anything like the strict holism that holds for entangled quantum particles.

There are many other distinctions and applications of holism that are inappropriate to try to review here. An extensive survey, at least in the philosophy of mind and in the philosophy of physics, is to be found in [20].

5 Conclusions

A set of observables is often considered as manifesting quantum effects if a joint probability distribution modeling its outcomes does not exist [40, 21]. However, in addition to such quantum effects, quantum theory has plenty of characteristics that, albeit reproducible by classical theory, are odd or puzzling. Here what we mean as a "classical theory" are local realistic hidden-variable theories, the type shown to be incompatible with some of the predictions of quantum theory [8].

A well-known example is the double-slit experiment. In his Lectures on Physics, Feynman said the double slit is "a phenomenon which is impossible, absolutely impossible to explain in any classical way, and which has in it the heart of quantum mechanics." However, way before Feynman's book was published, de Broglie, and later on, David Bohm, provided precisely such a model: the pilot wave interpretation. Though Bohm's model is non-local for two or more particles, it is still local and realistic for a single particle going through two slits. But Feynman's point is still valid, as the double-slit requires additional assumptions and a distinct ontology from the classical particle, at least in Bohm's model.

In this paper, we presented an example of a quantum system of observables that exhibit a classical but strange behavior. Namely, we showed that for a particular system of N entangled spin-1/2 particles, the values of spin present a strong type of holism. In our example, the product of N spins is deterministic. However, the product of any of its parts, $N-1$, $N-2$, $N-3$, and so on, is random. In other words, the whole behaves deterministically, whereas any of its parts behave randomly. As far as we know, no classical system exhibits such type of holism, excepted for contrived examples constructed as curiosities in probability textbooks.

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Appendix

As much of the mathematics used here is not widely known outside of physics, we briefly revise the main mathematical apparatus behind our quantum mechanical computations. We do so by showing in some detail examples of spin-1/2 systems. This Appendix is far from complete, and the interested reader should refer to one of the standard quantum mechanics textbooks (e.g., Cohen-Tannoudji et al. [14] or Sakurai [37]).

A quantum mechanical system is described by a state vector $|\psi\rangle \in \mathcal{H}$, where $\mathcal{H}$ is a Hilbert space, and any physical observable quantity is associated with a linear Hermitian operator on $\mathcal{H}$. A Hilbert space is a vector-space that is complete (i.e., all Cauchy sequences of vectors converge to a vector in $\mathcal{H}$) and has an inner product.

An observable is represented by a linear Hermitian operator acting on vectors in $\mathcal{H}$. Experiments that measure an observable $\hat{O}$ (operators in $\mathcal{H}$ are denoted here by capital letters with a hat over them) will have as allowed outcomes only the eigenvalues of the $\hat{O}$. Given that $\hat{O}$ is Hermitian, its eigenvalues are orthogonal and we can decompose a state $|\psi\rangle$ into the orthonormal basis of $\hat{O}$, i.e.,

$$|\psi\rangle = \sum_{i=1}^{N} c_i |i\rangle,$$

with $\hat{O}|i\rangle = o_i|i\rangle$, and o_i the set of eigenvalues of $\hat{O}$ (for the present discussion, the issue of completeness of operators will be ignored, as well as degeneracy, as they will be of no importance). The probability of observing the value o_i if a measurement of the observable $\hat{O}$ is performed is given by Born's rule as $P(o_i) = |c_i|^2$. Therefore, if follows at once that if $|\psi\rangle$ is an eigenvector of $\hat{O}$ with eigenvalue o_j, a measurement of $\hat{O}$ will have as outcome the value o_j with certainty.

Before we proceed any further, let us see the example of a single particle system where the particle has spin-1/2. If we are only interested in the spin observables, the Hilbert space for this particle has two dimensions. Experimentally, we observe that if we measure spin in the z-direction, we observe only two possible values: 1 or -1 (using a system of units where $\hbar = 2$). Mathematically, we would represent this experiment in the following way.

Let $\mathcal{H}_{1/2}$ be a two-dimensional complex Hilbert space, $|\psi\rangle$ be a vector in this space, i.e., $|\psi\rangle \in \mathcal{H}_{1/2}$, $\hat{S}_z$, $\hat{S}_x$, and $\hat{S}_y$ be linear operators in $\mathcal{H}_{1/2}$ representing measurements of spin in the z-, x-, and y-directions, respectively. If we represent the eigenvectors of $\hat{S}_z$ as $\hat{S}_z|+\rangle = |+\rangle$ and $\hat{S}_z|-\rangle = -|-\rangle$, then it follows from the spin-operator algebra that $\hat{S}_x|+\rangle = |-\rangle$, $\hat{S}_x|-\rangle = |+\rangle$,

$\hat{S}_y|+\rangle = -i|-\rangle$ and $\hat{S}_y|-\rangle = i|+\rangle$. From the actions of the spin operators above, it is easy to see that an eigenstate of $\hat{S}_z$ is not an eigenstate of $\hat{S}_x$ and *vice-versa.* For instance, the state $|+\rangle_x = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$ is an eigenstate of $\hat{S}_x$ with eigenvalue 1, since

$$\begin{aligned}
\hat{S}_x|+\rangle_x &= \hat{S}_x\frac{1}{\sqrt{2}}(|+\rangle + |-\rangle) \\
&= \frac{1}{\sqrt{2}}\left(\hat{S}_x|+\rangle + \hat{S}_x|-\rangle\right) \\
&= \frac{1}{\sqrt{2}}\left(\frac{1}{2}|-\rangle + \frac{1}{2}|+\rangle\right) \\
&= \frac{1}{2}\left[\frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)\right].
\end{aligned}$$

However, $\hat{S}_z|+\rangle_x \neq c|+\rangle_x$, where c is any complex number, and in fact we see that $|+\rangle_x$ has an orthornormal decomposition in the basis of $\hat{S}_z$ that yields components whose squares are equal to $\left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$. Therefore, according to Born's rule stated above, the probabilities of measuring 1 or -1 in the z-direction for state $|+\rangle_x$ are both 1/2.

Multiple-particle systems can be represented in quantum mechanics in essentially the same way as single particle systems, with the difference that the Hilbert space will have a higher dimensionality. For example, imagine that we want to represent two particles with spin-1/2. Each particle is described by its own Hilbert space, say $\mathcal{H}_{1/2,1}$ and $\mathcal{H}_{1/2,2}$, with the properties and observables satisfying the same rules as the single-particle system.

To describe a system composed of these two particles, we use a new Hilbert space $\mathcal{H}$ that is the tensor-product of the two Hilbert spaces, $\mathcal{H} = \mathcal{H}_{1/2,1} \otimes \mathcal{H}_{1/2,2}$. In the same way, we can represent experiments on both particles as observables that are the tensor product of observables on individual particles. For example, the observable in $\mathcal{H}$ that represents a measurement of spin in the z-direction for particle 1 and a measurement in the x-direction for particle 2 is the observable $\hat{S}_{z,1} \otimes \hat{S}_{x,2}$, where $\hat{S}_{z,1}$ is the z-direction spin observable acting on $\mathcal{H}_{1/2,1}$ and $\hat{S}_{x,2}$ is the x-direction spin observable acting on $\mathcal{H}_{1/2,2}$.

A state is represented as a vector in $\mathcal{H}$, and we can always write any state in terms of the base of tensor products of the subspaces $\mathcal{H}_{1/2,1}$ and $\mathcal{H}_{1/2,2}$. So, if $\{|+\rangle_1, |-\rangle_1\}$ and $\{|+\rangle_2, |-\rangle_2\}$ form a basis for $\mathcal{H}_{1/2,1}$ and $\mathcal{H}_{1/2,2}$, then

$$\{|+\rangle_1 \otimes |+\rangle_2, |-\rangle_1 \otimes |-\rangle_2, |+\rangle_1 \otimes |-\rangle_2, |-\rangle_1 \otimes |+\rangle_2\}$$

forms a basis for $\mathcal{H}$. Notice that, for example, state $|+\rangle_1 \otimes |-\rangle_2$ has a simple meaning: it is the state where if we measure spin in the z direction for both particles 1 and 2 we obtain +1 and −1, respectively. To confirm this interpretation we can apply the z-spin observable to both particles, i.e., $\hat{S}_{z,1} \otimes \hat{S}_{z,2}$, and verity that $|+\rangle_1 \otimes |-\rangle_2$ is indeed an eigenvector of this operator. The generalization to N particles is straightforward from the two particles.

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